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Corner transfer matrices and quantum affine algebras

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In memory of Roger Richardson

Abstract. We consider corner transfer matrices of the six-vertex model and of its spin-$l/2$ generalizations, truncated to a finite lattice, in the anti-ferromagnetic regime. We show that, in the sense of finite size approximation, the low-lying eigenvectors approximate weight vectors of a level-l highest weight $U_q\left(\widehat{sl}_2\right)$ module, whilst the truncated corner transfer matrices approximate the role of a derivation operator. Our method is to first use the theory of the crystal base to prove the result exactly at $q = 0$, and then extend to $q \neq 0$. We also consider the relationship between truncated corner transfer matrices, the vertex operator construction of Frenkel and Reshetikhin, and Baxter's original argument for the properties of CTMs.

1. Introduction

Corner transfer matrices (CTM), invented by Baxter [1,2], have proved to be an effective method for the evaluation of one-point functions in exactly solved lattice models of statistical mechanics, starting with the eight-vertex model [1], followed by the hard hexagon model [3] and its generalizations [4]. In the latter work, configuration sums appeared which were identified as characters of Virasoro algebras [5] and in particular those of the discrete minimal series [6]. These mysterious connections led to the discovery of hierarchies of solvable lattice models in two dimensions [7,8] and of many beautiful connections with infinite dimensional algebras [9–11].

Most of these connections were at the level of characters, weight-space multiplicities and branching coefficients. This is sufficient for the calculation of one-point functions since the CTMs of 'interaction around a face' models are constructed so that the diagonalizing transformation commutes with the state variable at the centre site. The calculations become thereby combinatorial and lead naturally to the use of q-series [12]. But it was also observed that the multiplicities of the CTM eigenvalues of the six-vertex model and its generalizations are equal to the weight-space multiplicities of irreducible highest-weight representations of certain affine Lie algebras [11]. Moreover, the actual configurations used to label the zero-temperature eigenvectors of the CTM also label the crystal base vectors [13,14] of the corresponding representations [15–17]. This is a vital clue, since the crystal base theory, which is the $q = 0$ limit of the quantum deformation of the commutators, contains complete information about the action of the Chevalley generators even when $q \neq 0$. If one wants to employ CTMs beyond the calculation of local height probabilities, precise information about the eigenvectors and their algebraic properties is needed, and the key is the use of quantum affine algebras.

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The six-vertex model is related to the quantum affine algebra $U_q(\widehat{sl_2})$ of Drinfeld and Jimbo [18, 19], since its Boltzmann weights form the R-matrix which intertwines tensor products of two-dimensional representations of $U_q'(\widehat{sl_2})$. A recent paper by Foda and Miwa [20] on the six-vertex CTM, and its connection with this algebra, has been the starting point for new and rapid progress in a number of different directions. (It should be noted that this work is in the anti-ferromagnetic regime—equivalent to regime III for the ABF models.) Foda and Miwa showed that the eigenvectors of the six-vertex CTM may be identified with the basic (level-1) representations of $U_q(\widehat{sl_2})$, and the CTM with a derivation operator of that algebra. Stated more simply, the quantum affine algebra provides raising and lowering operators for the CTM, and the importance of this cannot be overstated. For the XXZ model itself, this identification has been used to diagonalize the XXZ Hamiltonian for an infinite chain [21], and to obtain expressions for the n-point correlation functions [22], using the powerful machinery of representation theory and vertex operators [23].

Under the natural assumption that these results carry over to models associated with higher-spin representations of $U_q(\widehat{sl_2})$ (which we shall demonstrate in this paper), constructions of the state spaces have been given for higher spin [24] and for the RSOS models [25]. The latter paper uses a procedure somewhat analogous to the Goddard–Kent–Olive coset construction of the discrete minimal series of Virasoro algebras [26]. Thus the occurrence of Virasoro characters comes as no surprise, even though the GKO construction does not carry through to the quantum affine case and there is no obvious q-deformed analogue of the Virasoro algebra.

These developments highlight the importance of further study of the precise connections between CTMs and quantum affine algebras. This is the object of the present paper. The main statement of [20] is a conjecture for two reasons: (i) it assumes that the anti-ferromagnetic CTM Hamiltonian can be renormalized in the thermodynamic limit to produce a well-defined infinite-dimensional operator acting on a semi-infinite tensor product of spin one-half $U_q'(\widehat{sl_2})$ modules, and (ii) the necessary expansions are checked only to the first few orders in a perturbative expansion. These difficulties are pointed out in the cited paper, as are the technical problems preventing a proof that renormalization is possible to every order in a perturbative expansion.

In these circumstances it is natural to think in terms of low-temperature expansion using increasingly large but finite systems. Near the zero-temperature limit, such expansions are dominated by terms which differ from the ground state at only a finite number of sites, linked to a reference site by the exact Hamiltonian. We use this perturbative approach: our method is to truncate the semi-infinite spin chain on which the CTMs operate to a chain of length N, and then investigate the properties of these truncated operators as functions of q and N. The theory of the crystal base (the $q = 0$ limit) of infinite-dimensional highest-weight modules has already been investigated using this approach by Jimbo *et al* [16]. We recall some necessary details of their work in section 3, and present some new results. Here we point to a most remarkable feature. The basic object is the tensor product of N copies of a level-0 module. As such, it must retain the level zero. Notwithstanding, it is shown in [16] that a certain subset of its crystal base vectors serves as a subset of the crystal base of a level-l module. More recent investigations [27] demonstrate that this is intimately connected with the vertex operator construction of Frenkel and Reshetikhin [23]. The truncation to finite dimension allows expansion in powers of the deformation parameter q about $q = 0$ without problems of infinite renormalization. So we first compare the spectra of the truncated CTMs with the weight vectors and grading levels of the standard modules at $q = 0$. The identification thus made—which is exact—is then extended to $q \neq 0$, working always mod q^L for some integer L which increases linearly with N.

The necessity for a new approach becomes apparent if one attempts to find the eigenvectors of the CTMs directly in the $N \to \infty$ limit. One can try to expand in terms of the infinite set of paths $\mathcal{P}(\Lambda)$—for example, such an expansion is considered in some detail in [20]. It becomes clear from that paper that the coefficients of the paths are not normalizable ℓ_2 sequences. The point is that the anti-ferromagnetic ground state is 'infinitely far removed' from the bare states used in the expansion and an infinite renormalization is required. This is fundamentally different from the ferromagnetic regime, for which eigenvectors may be constructed as ℓ_2 sequences [28, 29], because the ground state is a single path. Once the anti-ferromagnetic eigenvectors are identified with weight vectors of standard $U_q\left(\widehat{sl_2}\right)$ modules, representation theory provides a unique inner product which respects the algebra action and for which $\langle u_\Lambda | u_\Lambda \rangle = 1$, and the normalization problem is resolved. What matters is that any q-expansion which arises from a low-temperature expansion should be identical to that obtained from representation theory, once the appropriate identifications are made.

Before embarking on the details of the present work, we briefly discuss the equivalent problem for the Ising model. This differs in one important respect: there is a simple technique for diagonalizing the finite-size CTM (see [28] for a discussion of the CTM, rather than its Hamiltonian generator). So the commutation relations, on which the solution is founded in both cases, are exact for the Ising case even with finite N. Still, one cannot give tractable general expressions for the excitation energies except when $N \to \infty$. Moreover, the structure of the ground state is exceedingly complicated. It is created from the 'bare vacuum' by applying the product of all N fermion annihilation operators—a simple highest-weight formula—but it should not be forgotten that this ground state is also not normalizable as an ℓ_2 sequence. The only new feature in the present case is that we do not have exact linear algebra formulae for the derivation of the quantum algebra since there is no derivation for finite N.

In both cases the CTM is used via a process of identification which is related to a non-uniform convergence property. To extract the essential properties of the eigenstates, one takes the limit $N \to \infty$ *before* using them [30]. The actual ground-state energy does not enter into any calculation—it is just part of an infinite renormalization. In the Ising case this does not seem mysterious since the fermion operators are so simple to understand, and one can work with the finite system up to any convenient point in the calculation. In the quantum algebra case, one must make contact with representation theory much more quickly. An important ingredient in this is the vertex operator construction, which gives a description of the local properties of the eigenvectors using the eigenvectors themselves.

The plan of this paper, and the main results, are as follows. In section 2 we introduce the necessary definitions of the quantum affine algebras. We also define the truncated CTMs and derive commutation relations from which their properties follow. These commutation relations are obtained for arbitrary spin, thus generalizing [20], which is restricted to spin one-half, and explaining how the commutation relations emanate from the intertwining property of R matrices. Section 3 uses the theory of the crystal base to prove that the eigenvectors of the truncated CTMs, at $q = 0$, are the crystal base vectors of the standard $U_q\left(\widehat{sl_2}\right)$ modules up to grading level $\approx N/2$, and that the eigenvalues provide the grading. We also resolve a problem from [16] about the relationship between the action of the modified Chevalley generators in the two crystal bases ('direction of the arrows'). These results are exact.

In section 4 we extend considerations to $q \neq 0$. We show that in this case the eigenvectors are weight vectors, $\bmod q^L$, where $L \to \infty$ in the limit of an infinite chain. This depends on more than just the basic commutation relations: crystal base theory is crucial for showing that the eigenvectors are in one-to-one correspondence with the weight

vectors at each grading level. Without this, the interpretation of physical problems using representation theory would be unclear: even though the eigenvectors approximate weight vectors: the question of completeness would remain. For precisely this reason we consider in section 5 the relationship between the eigenvectors of two truncated CTMs on chains whose lengths differ by one. This relationship turns out to be equivalent to the vertex operator construction, again mod q^L.

Section 6 is restricted to the six-vertex case. We revisit Baxter's original argument [31], whereby he found the remarkable exponentiation property of CTMs via arguments about the ground state of the corresponding spin chain. The same anti-ferromagnetic XXZ spin-chain ground state was constructed, using representation theory, in a recent work [21]. We show that the two arguments are equivalent—perhaps not surprising in retrospect, but nevertheless illuminating. Section 7 contains some concluding comments, particularly related to the calculation of physical quantities.

2. The algebra $U_q(\widehat{sl_2})$ and the CTM as its derivation

We are concerned with models defined on spin chains which act on tensor products whose components (position in the tensor product) correspond to the sites of the chain. For the quantum algebras we follow Jimbo [32]. $U_q'(\widehat{sl_2})$ is generated by e_i, f_i, $t_i = q^{h_i}$, $(i = 0, 1)$, which satisfy the defining relations

$$\begin{aligned}
&t_i e_j = q^{A_{ij}} e_j t_i \qquad t_i f_j = q^{-A_{ij}} f_j t_i \\
&t_i t_j = t_j t_i \qquad [e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}} \\
&e_i^3 e_j - (q^2 + 1 + q^{-2}) e_i^2 e_j e_i + (q^2 + 1 + q^{-2}) e_i e_j e_i^2 - e_j e_i^3 = 0 \quad (i \neq j) \\
&f_i^3 f_j - (q^2 + 1 + q^{-2}) f_i^2 f_j f_i + (q^2 + 1 + q^{-2}) f_i f_j f_i^2 - f_j f_i^3 = 0 \quad (i \neq j).
\end{aligned} \tag{2.1}$$

where A_{ij} is the generalized Cartan matrix for the affine Lie algebra $\widehat{sl_2}$,

$$A_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{2.2}$$

The co-multiplication for the Hopf algebra structure is defined by

$$\begin{aligned}
&\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i \\
&\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i \\
&\Delta(t_i) = t_i \otimes t_i.
\end{aligned} \tag{2.3}$$

and the formula for the antipode is

$$a(e_i) = -t_i^{-1} e_i \qquad a(f_i) = -f_i t_i \qquad a(t_i) = t_i^{-1}. \tag{2.4}$$

To obtain the full quantum affine algebra $U_q(\widehat{sl_2})$, we need to add the generator q^d. d is the derivation, and it satisfies the relations

$$[d, e_i] = \delta_{i,0} e_i, \qquad [d, f_i] = -\delta_{i,0} f_i. \tag{2.5}$$

Our notations for $\widehat{sl_2}$ are as follows. The Cartan subalgebra $\mathfrak{h} = \text{span}\{h_0, h_1, d\}$ and α_0, α_1 are the roots. They are related to the fundamental weights by $\alpha_0 = 2\Lambda_0 - 2\Lambda_1 + \delta$, $\alpha_1 = 2\Lambda_1 - 2\Lambda_0$; we also write $\rho = \Lambda_0 + \Lambda_1$. The invariant form on $\mathfrak{h}^*$ is given by $2(\Lambda_i, \Lambda_j) = \delta_{1i}\delta_{1j}$, $(\Lambda_i, \delta) = 1$, $(\delta, \delta) = 0$. The weight lattice and its dual are $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta$ and $P^* = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \mathbb{Z}d$, with $\langle \Lambda_i, h_j \rangle = \delta_{ij}$, $\langle \Lambda_i, d \rangle = 0$, $\langle \delta, h_i \rangle = 0$ and $\langle \delta, d \rangle = 1$. We identify P^* with a subset of P via $(\ ,\ )$, so that $\alpha_i = h_i$ and $2\rho = 4d + h_1$.

The representations we use are built on the basic $(l+1)$-dimensional modules V_l of $U_q(sl_2)$. We regard the latter as a subalgebra of $U_q'(\widehat{sl_2})$ by the identification $e = e_1$, $f = f_1$, $t = t_1$. Then the basic $U_q(sl_2)$ modules V_l, with weight vectors v_k, $(k = 0, \ldots, l)$, are defined by the actions

$$\begin{aligned} ev_k &= [k]v_{k-1} \\ fv_k &= [l-k]v_{k+1} \\ tv_k &= q^{l-2k}v_k \end{aligned} \tag{2.6}$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$ and $v_k = 0$ if $k < 0$ or $k > l$. From these are constructed irreducible finite-dimensional representations $V(l, x)$ of $U_q'(\widehat{sl_2})$, with multiplicative spectral parameter x, obtained by the identification

$$e_0 \to xf \qquad f_0 \to x^{-1}e \qquad t_0 \to t^{-1}e_1 \to e \qquad f_1 \to f \qquad t_1 \to t. \tag{2.7}$$

The R-matrices $\check{R}(x, y)$ which satisfy the Yang–Baxter equations with trigonometric parametrisation are intertwiners of tensor products $V(l, x) \otimes V(l, y)$ and $V(l, y) \otimes V(l, x)$. Details may be found in [32]; here we recall the facts that are of immediate interest. The intertwining property entails $\check{R}(x, y)\Delta(a) = \Delta(a)\check{R}(x, y)$, for all $a \in U_q'(\widehat{sl_2})$. Together with a choice of normalization, the intertwining property uniquely determines $\check{R}(x, y)$. The formula is given in [33] in terms of the Clebsch–Gordan decomposition of the tensor product $V_l \otimes V_l$ as $U_q(sl_2)$ modules: $V_l \otimes V_l \simeq V_{2l} \oplus V_{2l-2} \oplus \cdots \oplus V_0$. Let P_r denote the projection operator into the component V_{2l-2r} of this decomposition. Then

$$\check{R}(x, y) = \sum_{r=0}^{l} \prod_{s=1}^{r} \left(\frac{x - yq^{2l-2s+2}}{y - xq^{2l-2s+2}} \right) P_r. \tag{2.8}$$

Corresponding to such a matrix $\check{R}(x, y)$, spin chains are built from two-site operators obtained by expansion around $x = y$:

$$\check{R}(x, y) = 1 + uH_j + \cdots \qquad x/y = e^u \qquad u \to 0. \tag{2.9}$$

We have attached a subscript j to H_j in anticipation of the fact that it will operate at the positions $j+1$, j of the chain. (We follow the convention of [29] in numbering the sites from right to left.) For this reason, we shall henceforth attach a second subscript to the generators e_i, f_i, t_i, writing $e_{i,j}$, $f_{i,j}$, $t_{i,j}$. Using the notation that $P_{r,j}$ stands for the action of P_r on the tensor product $V_l \otimes V_l$ at sites $j+1$, j we have

$$H_j = \sum_{r=1}^{l} \sum_{s=1}^{r} \left(\frac{1 + q^{2l-2s+2}}{1 - q^{2l-2s+2}} \right) P_{r,j}. \tag{2.10}$$

We need the commutation relations between H_j and the coproduct of the generators e_i, f_i, onto sites $j+1$, j. For $\Delta(e_1)$, $\Delta(f_1)$ this is trivial: the use of the projectors P_r in the construction of $\check{R}(x, y)$ comes from the fact that the intertwining action applies in particular to the subalgebra $U_q(\mathfrak{sl}_2)$, whose action on the tensor product does not depend on the spectral parameters. Therefore

$$\begin{aligned}&[H_j, e_{1,j+1} \otimes 1_j + t_{1,j+1} \otimes e_{1,j}] = 0,\\ &[H_j, f_{1,j+1} \otimes t_{1,j}^{-1} + 1_{j+1} \otimes f_{1,j}] = 0.\end{aligned} \tag{2.11}$$

In fact, H_j is just a polynomial in the Casimir operator C of $U_q(\mathfrak{sl}_2)$ acting on adjacent pairs of sites. We take for C the definition

$$C = qt + q^{-1}t^{-1} + (q - q^{-1})^2 fe \tag{2.12}$$

and note that it has the value $(q^{l+1} + q^{-l-1})$ on the basic module V_l. Writing C_j to stand for $\Delta(C)$ acting on the tensor product at sites $j+1$, j we have

$$H_j = \phi_l(C_j) \tag{2.13}$$

where $\phi_l(x)$ is a polynomial of degree l, determined from (2.10), which gives the relations

$$\phi_l(q^{2l-2r+1} + q^{-2l+2r-1}) = \sum_{s=1}^{r} \left(\frac{1 + q^{2l-2s+2}}{1 - q^{2l-2s+2}}\right) \qquad r = 0, \ldots, l. \tag{2.14}$$

The intertwining conditions for e_0, f_0, which are the nub of the question of relating CTMs to the derivations of $U_q(\widehat{\mathfrak{sl}_2})$, may be written explicitly as [32]

$$\begin{aligned}&\check{R}(x, y)(xf \otimes 1 + t^{-1} \otimes yf) = (yf \otimes 1 + t^{-1} \otimes xf)\check{R}(x, y)\\ &\check{R}(x, y)(x^{-1}e \otimes t + 1 \otimes y^{-1}e) = (y^{-1}e \otimes t + 1 \otimes x^{-1}e)\check{R}(x, y).\end{aligned} \tag{2.15}$$

Making the substitution $x = ye^u$ into these equations and using the definition of the H_j, we find the commutation relations

$$\begin{aligned}&[H_j, e_{0,j+1} \otimes 1_j + t_{0,j+1} \otimes e_{0,j}] = -e_{0,j+1} \otimes 1_j + t_{0,j+1} \otimes e_{0,j},\\ &[H_j, f_{0,j+1} \otimes t_{0,j}^{-1} + 1_{j+1} \otimes f_{0,j}] = f_{0,j+1} \otimes t_{0,j}^{-1} - 1_{j+1} \otimes f_{0,j}.\end{aligned} \tag{2.16}$$

Consider now the spin chain $H = \sum_{j=1}^{N-1} \alpha_j H_j$, with arbitrary coefficients α_j, which acts on the N-fold tensor product $V_l^{\otimes N}$. We shall write $\hat{e}_i$, $\hat{f}_i$, $\hat{t}_i$ for the iterated co-product of the generators; explicitly

$$\begin{aligned}\hat{e}_i &= \sum_{j=1}^{N} t_{i,N} \otimes \cdots \otimes t_{i,j+1} \otimes e_{i,j} \otimes 1_{j-1} \otimes \cdots \otimes 1_1,\\ \hat{f}_i &= \sum_{j=1}^{N} 1_N \otimes \cdots \otimes 1_{j+1} \otimes f_{i,j} \otimes t_{i,j-1}^{-1} \otimes \cdots \otimes t_{i,1}^{-1}\\ \hat{t}_i &= t_{i,N} \otimes \cdots \otimes t_{i,1}.\end{aligned} \tag{2.17}$$

Equations (2.811) show that the chain H is $U_q(\mathfrak{sl}_2)$ invariant, even with arbitrary coefficients α_j. That is,

$$[H, \hat{e}_1] = 0 \qquad [H, \hat{f}_1] = 0. \tag{2.18}$$

For $i = 0$, matters are slightly more complicated. To obtain the CTM generator, we choose $\alpha_j = j$, and

$$H_{\mathrm{CTM},N} = \sum_{j=1}^{N-1} j H_j. \tag{2.19}$$

Then most of the terms in (2.16) sum to the definitions of $\hat{e}_0$ and, $\hat{f}_0$, so that we find

$$\begin{aligned} [H_{\mathrm{CTM},N}, \hat{e}_0] &= \hat{e}_0 - N e_{0,N} \otimes 1_{N-1} \cdots \otimes 1_1 \\ [H_{\mathrm{CTM},N}, \hat{f}_0] &= -\hat{f}_0 + N f_{0,N} \otimes t_{0,N-1}^{-1} \otimes \cdots \otimes t_{0,1}^{-1}. \end{aligned} \tag{2.20}$$

This is a vital result: provided there is a satisfactory way to interpret the infinite N limit in a regime where the boundary terms on the RHS of (2.20) may be neglected, we recover the result that $H_{\mathrm{CTM},\infty}$ acts as the derivation in the algebra $U_q(\widehat{\mathfrak{sl}_2})$, verifying the relations (2.5). We emphasise that our use of (2.20) at the crystal base does *not* involve the boundary term, since we restrict the paths to those that satisfy the boundary conditions when $q = 0$. Also note that although the boundary term diverges linearly as $N \to \infty$ (like all the terms in $H_{\mathrm{CTM},N}$), this has no effect on our arguments in sections 4 and 5: $Nq^N \to 0$ just as well as q^N does as $N \to \infty$.

For the six-vertex case ($l = 1$), the two site operators H_j of (2.9) are (to within an additive constant)

$$\frac{1}{q - q^{-1}} \left[\frac{1}{2}(\sigma_{j+1}^x \sigma_j^x + \sigma_{j+1}^y \sigma_j^y) + \frac{(q + q^{-1})}{4} \sigma_{j+1}^z \sigma_j^z + \frac{(q - q^{-1})}{4}(\sigma_{j+1}^z - \sigma_j^z) \right] \tag{2.21}$$

where σ_j^x, σ_j^y, σ_j^z are the usual Pauli spin matrices. The XXZ four-fermion coupling parameter Δ is related to the quantum algebra deformation q by

$$\Delta = (q + q^{-1})/2 \tag{2.22}$$

and $-1 < q < 0$ in the anti-ferromagnetic regime, $\Delta < -1$. The term involving $(\sigma_{j+1}^z - \sigma_j^z)$ is the well known boundary term which is responsible for $U_q(\mathfrak{sl}_2)$ symmetry in XXZ spin chains [34, 35]. It gives an additional contribution to the more usual six-vertex CTM generator [31]

$$H_{\mathrm{XXZ},N} = \frac{1}{2\sqrt{\Delta^2 - 1}} \sum_{j=1}^{N-1} j(\sigma_{j+1}^x \sigma_j^x + \sigma_{j+1}^y \sigma_j^y + \Delta \sigma_{j+1}^z \sigma_j^z). \tag{2.23}$$

The total spin operator $S = \sum_{j=1}^{N} \sigma_j^z$ commutes with both $H_{\mathrm{CTM},N}$ and $H_{\mathrm{XXZ},N}$, but there is in general no simple relation of the form $aH_{\mathrm{CTM},N} - bH_{\mathrm{XXZ},N} = S$. However, as observed by Foda and Miwa [20], we do have such a relationship in the anti-ferromagnetic regime,

provided that suitable renormalized operators may be defined in the thermodynamic limit, namely,

$$H_{\mathrm{XXZ},\infty} = 2H_{\mathrm{CTM},\infty} + S \qquad 2S = \sum_{k=1}^{\infty} \sigma_j^z. \tag{2.24}$$

The commutation relations for $H_{\mathrm{XXZ},\infty}$ are

$$[H_{\mathrm{XXZ},\infty}, \hat{e}_i] = \hat{e}_i \qquad [H_{\mathrm{XXZ},\infty}, \hat{f}_i] = -\hat{f}_i. \tag{2.25}$$

So $H_{\mathrm{XXZ},\infty}$ acts as the derivation in the principal grading whereas $H_{\mathrm{CTM},\infty} = d$ is the derivation in the homogeneous grading. Some illustrative numerical computations for the six-vertex case are reported in a preliminary version of this paper [36].

For real q, $|q| < 1$, the coefficient of $P_{r,j}$ in (2.10) is non-negative for all r. It follows that H_j and therefore $H_{\mathrm{CTM},N}$ are non-negative operators. Consider the action of $H_{\mathrm{CTM},N}$ on either of the vectors $v_0 \otimes \cdots \otimes v_0$ or $v_l \otimes \cdots \otimes v_l$. Since $v_0 \otimes v_0$ and $v_l \otimes v_l$ are the highest- and lowest-weight vectors in $V_l \otimes V_l$, H_j is zero on them. Therefore $H_{\mathrm{CTM},N}$ has a zero eigenvalue and we have found two of the zero eigenvectors: $v_0 \otimes \cdots \otimes v_0$ and $v_l \otimes \cdots \otimes v_l$. But these are also highest- and lowest-weight vectors in $V_l^{\otimes N}$, in fact of the largest $U_q(\mathfrak{sl}_2)$ multiplet of dimension $lN + 1$. By the $U_q(\mathfrak{sl}_2)$ symmetry of $H_{\mathrm{CTM},N}$, this multiplet of basic ferromagnetic states is the zero eigenspace of $H_{\mathrm{CTM},N}$. The maximum eigenvalue of $H_{\mathrm{CTM},N}$ is an anti-ferromagnetic ground state, and the various choices we have made in the definitions ensure that this becomes a highest-weight vector $|\mathrm{hwv}\rangle$ of some module $V(\Lambda)$ in the limit $N \to \infty$.

3. The crystal base

The crystal base is the $q \to 0$ limit of the theory. The essential simplification is that the basis vectors of tensor products of irreducible modules are just the elementary tensor products. In this limit the operators H_j take the simple form

$$H_j^0 = \sum_{r=1}^{l} r P_{r,j}, \qquad q \to 0 \tag{3.1}$$

and $H_{\mathrm{CTM},N}^0 = \left[H_{\mathrm{CTM},N}\right]_{q=0}$ is diagonal, as an operator in $V_l^{\otimes N}$, on the basis vectors $u_{i_N} \otimes \cdots \otimes u_{i_1}$. The action of the generators e_i, f_i, t_i, becomes undefined in this limit, but Kashiwara showed how to define modified Chevalley generators $\tilde{e}_i$, $\tilde{f}_i$ so that their action on the crystal base vectors has an exact correspondence with the action of the actual generators [13, 14]. This correspondence is very powerful, and makes it possible to prove very general results from more combinatorial considerations at the crystal base.

For the module V_l, we may use the base vectors $\{v_k\}$ also as the (upper) crystal base vectors, but we shall denote them as $\{b_k\}$ when so doing. (Strictly speaking, b_k represent residue classes modulo the crystal lattice.) We also use the convention that B_l, $B(\Lambda)$, etc denote the crystal base of modules V_l, $V(\Lambda)$. The action of the modified Chevalley generators on B_l is given by

$$\begin{aligned} \tilde{e}_1 b_k &= b_{k-1} \\ \tilde{f}_1 b_k &= b_{k+1} \end{aligned} \tag{3.2}$$

with $b_k = 0$ if $k < 0$ or $k > l$. Also $\tilde{e}_0 = \tilde{f}_1$, $\tilde{f}_0 = \tilde{e}_1$. These actions may be represented as a coloured oriented graph in which each node is a crystal base vector, and the nodes are joined by arrows labelled by the 'colour' i. The direction of the arrow indicates the action of $\tilde{f}_i$.

$$
\begin{array}{ccc}
b_0 \otimes b_0' & \xrightarrow{1} & b_1 \otimes b_0' \\
\uparrow 0 & & \downarrow 1 \\
b_0 \otimes b_1' & \xleftarrow{0} & b_1 \otimes b_1'
\end{array}
\qquad
\begin{array}{ccccc}
b_0 \otimes b_0' & \xrightarrow{1} & b_1 \otimes b_0' & \xrightarrow{1} & b_2 \otimes b_0' \\
\uparrow 0 & & \uparrow 0 & & \downarrow 1 \\
b_0 \otimes b_1' & \xrightarrow{1} & b_1 \otimes b_1' & \xleftarrow{0} & b_2 \otimes b_1' \\
\uparrow 0 & & \downarrow 1 & & \downarrow 1 \\
b_0 \otimes b_2' & \xleftarrow{0} & b_1 \otimes b_2' & \xleftarrow{0} & b_2 \otimes b_2'
\end{array}
$$

Figure 1. Crystal graphs $B_l \otimes B_l$ for $l = 1, 2$, as $U_q'\left(\widehat{s\mathfrak{l}_2}\right)$ algebra.

The rule for the crystal graph of the tensor product of two modules V, V' is as follows. For each i, a crystal base vector b is part of a string through b spanning an irreducible $U_q(s\mathfrak{l}_2)$ module for colour i. Introduce length functions $l_i^{\pm}(b)$ for these strings with the meaning that $l_i^+(b)$ (respectively, $l_i^-(b)$) is the length of the part of the string produced by applying $\tilde{e}_i$ (respectively, $\tilde{f}_i$) to b. The crystal base $B \otimes B'$ is the set of products $b \otimes b'$. The action of $\Delta(\tilde{e}_i)$ on $b \otimes b'$ gives either the product $(\tilde{e}_i b) \otimes b'$ or $b \otimes (\tilde{e}_i b')$ (similarly for $\tilde{f}_i$). The actual result depends on the i-string lengths through b and b'. The rule is [32]

$$
\begin{aligned}
\tilde{e}_i(b \otimes b') &= \begin{cases} b \otimes \tilde{e}_i b' & l_i^-(b) < l_i^+(b') \\ \tilde{e}_i b \otimes b' & l_i^-(b) \geqslant l_i^+(b') \end{cases} \\
\tilde{f}_i(b \otimes b') &= \begin{cases} \tilde{f}_i b \otimes b' & l_i^-(b) > l_i^+(b') \\ b \otimes \tilde{f}_i b' & l_i^-(b) \leqslant l_i^+(b'). \end{cases}
\end{aligned}
\tag{3.3}
$$

Irreducible modules have crystal graphs which are connected so that crystal graphs of tensor products illustrate pictorially the irreducible pieces of the decomposition.

As an example relevant to the operators H_j of (4.1), the crystal graphs are shown in figure 1 for the tensor product $B_l \otimes B_l$, as $U_q'\left(\widehat{s\mathfrak{l}_2}\right)$ modules, for $l = 1, 2$. One sees clearly that the tensor product is irreducible as a $U_q'\left(\widehat{s\mathfrak{l}_2}\right)$ module but reducible as a $U_q(s\mathfrak{l}_2)$ module of either colour 0 or 1. H_j, which is diagonal for $q = 0$, takes different integer values on each irreducible component (connected subgraph) of the decomposition. A simple computation shows that the value $H(j, j')$ of H_j on $b_j \otimes b_{j'}$ is the energy function of [16]:

$$
H(j, j') = \begin{cases} j' & j + j' \geqslant l \\ l - j & j + j' \leqslant l\,. \end{cases}
\tag{3.4}
$$

Jimbo *et al* have considered the crystal bases of tensor products $V_l^{\otimes N}$ as finite-size approximations to the crystal base of the irreducible level-l modules of $U_q\left(\widehat{s\mathfrak{l}_2}\right)$ [16]. As we noted in the introduction, this seems somewhat paradoxical, since a finite tensor product of level-0 modules is necessarily of level 0. But there is no paradox provided one

notes carefully what is actually proved in [16]. Here we show that $H^0_{\mathrm{CTM},N}$ is a finite-size approximation to the derivation in this same construction. But first we recall the necessary results from [16]. Let Λ_0, Λ_1 be the fundamental weights of $U_q(\widehat{\mathrm{sl}_2})$. Then the highest weights of the irreducible level-l modules $\{k\Lambda_0 + (l-k)\Lambda_1 \mid 0 \leqslant k \leqslant l\}$, and they are in correspondence with the basis $\{v_k\}$ of V_l.

The first major result of [16] is the construction of a crystal base for the modules $V(\Lambda)$ as sets of semi-infinite paths $\mathcal{P}(\Lambda)$, $\{p_j \mid 0 \leqslant p_j \leqslant l,\ j \geqslant 1\}$. For a given highest weight Λ, the 'ground state' path $\bar{p}_\Lambda$ is defined by

$$\bar{p}_\Lambda = \{\bar{p}_j \mid j \geqslant 1\} \qquad \bar{p}_j = \begin{cases} k & \text{for } j \text{ odd} \\ l-k & \text{for } j \text{ even.} \end{cases} \tag{3.5}$$

Then the set $\mathcal{P}(\Lambda)$ consists of all paths which differ from the ground state path at only a finite number of places. $\mathcal{P}(\Lambda)$ is a crystal base for $V(\Lambda)$, and the weight of a path is

$$\mathrm{wt}(p) = \Lambda - \alpha_1 \sum_{j\geqslant 1}(p_j - \bar{p}_j) - \delta \sum_{j\geqslant 1} j(H(p_{j+1}, p_j) - H(\bar{p}_{j+1}, \bar{p}_j)). \tag{3.6}$$

The second major result of [16] is that the action of the modified Chevalley generators on any particular path is equivalently stated in terms of the action on the finite tensor product $V_l^{\otimes N}$ for sufficiently large N. (Specifically, on the subset of paths for which $p_N = \bar{p}_N$.) However, there is a problem that the roles of $\tilde{e}_i$ and $\tilde{f}_i$ are interchanged in this correspondence. This interchange is avoided here by numbering the sites of the chain from right to left. To see why, let φ denote the automorphism of $U_q'(\widehat{\mathrm{sl}_2})$ given by

$$\varphi(e_i) = f_i \qquad \varphi(f_i) = e_i \qquad \varphi(t_i) = t_i^{-1} \qquad (i = 0, 1). \tag{3.7}$$

and let $(\pi, V_l^{\otimes N})$ denote the representation of $U_q'(\widehat{\mathrm{sl}_2})$ on the tensor product $V_l^{\otimes N}$ which we have employed via the standard co-multiplication (2.3) and the module action (2.4). The map φ is not a Hopf-algebra homomorphism, and the $U_q'(\widehat{\mathrm{sl}_2})$ representation $\pi \circ \varphi$ is related to π by transposition,

$$\begin{aligned} (\pi\circ\varphi)(e_i)(v_{i_1}\otimes\cdots\otimes v_{i_N}) &= \pi(f_i)(v_{i_N}\otimes\cdots\otimes v_{i_1}) \\ (\pi\circ\varphi)(f_i)(v_{i_1}\otimes\cdots\otimes v_{i_N}) &= \pi(e_i)(v_{i_N}\otimes\cdots\otimes v_{i_1}). \end{aligned} \tag{3.8}$$

This result is also seen in the fact that the vertex operator construction, for highest-weight modules (rather than lowest-weight and/or dual modules), is an isomorphism of $B(\Lambda)\otimes B_l$ with $B(\Lambda')$, and the semi-infinite tail is to the left as the isomorphism is iterated [27].

We turn now to the connection between the spectrum of $H^0_{\mathrm{CTM},N}$ and the weights of paths in $\mathcal{P}(\Lambda)$. A simple calculation shows that the value of $H^0_{\mathrm{CTM},N}$, on the ground state vector in $B_l^{\otimes N}$ corresponding to $\bar{p}_\Lambda$, is

$$E_N(\bar{p}_\Lambda) = \begin{cases} N(N-2)l/4 + Nk/2 & \text{for } N \text{ even} \\ (N-1)^2 l/4 + (N-1)(l-k)/2 & \text{for } N \text{ odd.} \end{cases} \tag{3.9}$$

Therefore, $H^0_{\mathrm{CTM},N}$ attains its maximum value in $B_l^{\otimes N}$ on the highest-weight path of $B(l\Lambda_0)$ or $B(l\Lambda_1)$ according to whether N is even or odd. If we simply diagonalize $H_{\mathrm{CTM},N}$, without imposing boundary conditions, the low-lying eigenvectors approximate the weight vectors

of either $V(l\Lambda_0)$ or $V(l\Lambda_1)$. To obtain the weight vectors of all the level-l modules, we must impose the boundary condition $p_N = \bar{p}_N$ using the appropriate ground state path $\bar{p}_\Lambda$ (exactly as in [16]). That is, we define the subspace $W_\Lambda \subset V_l^{\otimes N}$ to be the span of vectors satisfying this boundary condition and we diagonalize the restriction of $H_{\mathrm{CTM},N}$ to W_Λ rather than the full matrix. Since $H^0_{\mathrm{CTM},N}$ is diagonal, the W_Λ are already invariant subspaces and there are no neglected terms when $q = 0$. Then we have the corresponding $B_\Lambda \subset B_l^{\otimes N}$, and it is trivial to show that $H^0_{\mathrm{CTM},N}$ attains its maximum value in B_Λ on the highest-weight path for $B(\Lambda)$, irrespective of whether N is even or odd.

Consider the set of paths in $\mathcal{P}(\Lambda)$ which agree with the ground-state path $\bar{p}_\Lambda$ at site N and identify them with the corresponding vectors in $B_\Lambda \subset B^{\otimes N}$. The vectors are of the form $b_{k_N} \otimes b_{k_{N-1}} \otimes \cdots \otimes b_{k_1}$. We may show that the commutation relations (2.28), restricted to B_Λ, are exact at the crystal base: $[H^0_{\mathrm{CTM},N}, \tilde{e}_0] = \tilde{e}_0$, $[H^0_{\mathrm{CTM},N}, \tilde{f}_0] = -\tilde{f}_0$. Equivalently, in this subspace $\tilde{e}_0$ and $\tilde{f}_0$ are raising and lowering operators for $H^0_{\mathrm{CTM},N}$. We give the argument for $\tilde{f}_0$. Its action is to change just one of the crystal base vectors, at position $j \neq N$. From the co-associativity of the co-product, we know that the action of $\tilde{f}_0$ on the adjacent pairs $b_{k_{j+1}} \otimes b_{k_j}$ and $b_{k_j} \otimes b_{k_{j-1}}$ is exactly the same as its action when the pair is embedded in the full tensor product $b_{k_N} \otimes b_{k_{N-1}} \otimes \cdots \otimes b_{k_1}$. That is,

$$\begin{aligned} \tilde{f}_0(b_{k_{j+1}} \otimes b_{k_j}) &= b_{k_{j+1}} \otimes \tilde{f}_0 b_{k_j} \\ \tilde{f}_0(b_{k_j} \otimes b_{k_{j-1}}) &= \tilde{f}_0 b_{k_j} \otimes b_{k_{j-1}} \end{aligned} \tag{3.10}$$

and the rules (4.3) give the conditions for this, in terms of the variables p_j, as $p_j + p_{j+1} \leqslant l$, $p_{j-1} + p_j > l$. Applying (4.4) we see that the energy of the state is decreased by 1 by the action of $\tilde{f}_0$, which was to be proved, and we have:

Proposition 1. *Let $\Lambda = k\Lambda_0 + (l-k)\Lambda_1$ and fix N. For the level-l $U_q(\widehat{\mathfrak{sl}_2})$ module $V(\Lambda)$, the weights (4.6) of those paths in $\mathcal{P}(\Lambda)$ which agree with the ground state path $\bar{p}_\Lambda$ at sites $j \geqslant N$ are correctly given if the energy term is replaced by $-\delta(E(p) - E(\bar{p}))$. Here $E(p)$ are the eigenvalues of $H^0_{\mathrm{CTM},N}$ diagonalized in the subspace B_Λ.*

We want to prove a version of this proposition which takes account of the grading of the standard modules,

$$V(\Lambda) = \bigoplus_{n=0}^{\infty} V_n \qquad V_n = \bigoplus_m V_{n,m} \tag{3.11}$$

where $V_{n,m}$ are weight spaces of weight $\Lambda - m\alpha_1 - n\delta$. The weight spaces of $B(\Lambda)$ are spanned by vectors generated by acting with multinomials $\tilde{f}_0^{m_j} \tilde{f}_1^{n_j} \cdots \tilde{f}_0^{m_1} \tilde{f}_1^{n_1}$ on the highest-weight vector $b_{k_N} \otimes \cdots \otimes b_{l-k} \otimes b_k$. The integers n_1 and m_j may be zero. As we perform each operation $\tilde{f}_i^m$, the left-most position in the tensor product at which a component differs from its value in the ground state vector will increase, but only by unity. So any crystal base vector in the crystal base B_n of V_n cannot differ from the ground state vector beyond site $2n$. Since we already know from [16] that the multiplicities of paths with any given weight are correct for sufficiently large N, we have:

Proposition 2. *Let $\Lambda = k\Lambda_0 + (l-k)\Lambda_1$ and fix integers M, N with $N > 2M + 1$. Let $E(p)$ be the eigenvalues of $H^0_{\mathrm{CTM},N}$ diagonalized in the subspace $B_\Lambda \subset B_l^{\otimes N}$. Select those eigenvectors for which $(E(\bar{p}) - E(p)) \leqslant M$. The corresponding set of paths forms part of the crystal base $B(\Lambda)$ of $V(\Lambda)$, specifically they span $\bigoplus_{n=0}^{M} B_n \subset B(\Lambda)$.*

4. Eigenvectors of CTMs as weight vectors

The main aim of this section is to give a precise sense in which the CTM provides a finite-size approximation to the derivation d of $U_q(\widehat{sl_2})$, and its eigenvectors to weight vectors. We shall show that the set of weight vectors with weights $\Lambda - m\alpha_1 - n\delta$, for arbitrary n may be approximated by diagonalizing $H_{\mathrm{CTM},N}$ on a sufficiently long chain with appropriate boundary conditions. The boundary conditions are applied by diagonalizing a restriction of $H_{\mathrm{CTM},N}$ to the subspace $W_\Lambda \subset V_l^{\otimes N}$ spanned by states which satisfy the boundary condition. Only the last two-site operator, H_{N-1}, is involved in this restriction. Its off-diagonal parts, which only connect vectors in different subspaces W_Λ, is discarded. The restriction of H_{N-1} to W_Λ is therefore diagonal, a fact which we need below. $H_{\mathrm{CTM},N}$ is just a finite matrix, and our sense of approximation is with respect to an inner product which we now define. Recall that $H_j = \phi_l(C_j)$, where $C_j = \Delta(C)$. Explicitly,

$$C_j = q(t_{j+1} \otimes t_j) + q^{-1}(t_{j+1}^{-1} \otimes t_j^{-1}) + (q - q^{-1})^2 \times (f_{j+1}e_{j+1} \otimes t_j^{-1} + t_{j+1} \otimes f_j e_j + e_{j+1} \otimes f_j + f_{j+1}t_{j+1} \otimes t_j^{-1} e_j). \tag{4.1}$$

For the six-vertex case, $H_{\mathrm{CTM},N}$ is a real symmetric matrix for real q if we simply declare the basis vectors v_k of (2.4) to be orthonormal and employ the induced inner product in $V_1^{\otimes N}$. For general l, we see from (5.4) that we could normalize the basis vectors v_k so that $\langle v_{k-1}|ev_k\rangle = \langle t^{-1}ev_k|v_{k-1}\rangle$ and $\langle v_k|fv_{k-1}\rangle = \langle ftv_{k-1}|v_k\rangle$ and this will symmetrise $H_{\mathrm{CTM},N}$. However, we choose not to do so: $\langle\ |\ \rangle$ is the standard inner product with the consequence that for $l > 1$, $H_{\mathrm{CTM},N}$ has distinct right and left eigenvectors which form a bi-orthogonal set. They become weight vectors of right and left modules as $N \to \infty$.

Proposition 3. Let Λ be a level-l dominant integral weight with corresponding ground state $\bar{p}_\Lambda$. Choose $M \in \mathbb{Z}$ and consider the eigenvectors of the chain $H_{\mathrm{CTM},N}$ diagonalized on $N \in \mathbb{Z}$, $N > M+1$ sites subject to the fixed boundary condition $p_N = (\bar{p}_\Lambda)_N$. Let $\{|\psi\rangle\}$ be the set of those eigenvectors which correspond, as $q \to 0$, to paths $p \in \mathcal{P}(\Lambda)$ agreeing with the ground state $\bar{p}_\Lambda$ at sites $j \geqslant M$. Then the commutation relations (2.3d) *are satisfied,* mod $q^{2N-2M-1}$, *on the subspace spanned by $\{|\psi\rangle\}$, with d replaced by $H_{\mathrm{CTM},N}$.*

Proof. Fix M and let S be the set of states defined in the proposition. For brevity we shall identify the elementary tensor product $v_{j_N} \otimes \cdots \otimes v_{j_1}$ with the path in $\mathcal{P}(\Lambda)$ which labels its components. Consider the commutation relation $[d, e_0] = e_0$. It is sufficient to show that

$$\langle\psi|([H_{\mathrm{CTM},N}, e_0] - e_0)\psi'\rangle = 0 \qquad (\mathrm{mod}\, q^{2N-2M-1}) \tag{4.2}$$

for any pair of states from S and any integer $N > M$. For this we use (2.20) for $H_{\mathrm{CTM},N-1}$ and treat the restriction of H_{N-1} separately. Remembering that the latter is diagonal, we get

$$\begin{aligned}\langle\psi|([H_{\mathrm{CTM},N}, e_0] - e_0)\psi'\rangle &= (N-1)\langle\psi|1_N \otimes e_{0,N-1} \otimes 1_{N-2} \cdots \otimes 1_1\psi'\rangle \\ &\quad + (N-1)\langle\psi|[H_{N-1}, 1_N \otimes e_{0,N-1} \otimes 1_{N-2} \cdots \otimes 1_1]\psi'\rangle.\end{aligned} \tag{4.3}$$

Next, we need to expand the states $|\psi\rangle$ in powers of q:

$$|\psi\rangle = \sum_{k=0}^{L} q^k |\psi^{(k)}\rangle + \mathrm{O}(q^{L+1}). \tag{4.4}$$

The overall normalization of H_j does not affect the expansion (4.4), nor is it relevant for the normalization of the states. Therefore we replace H_j by $H_j^0 + \epsilon H_j^1$, where H_j^0 and H_j^1 are independent of q with $\epsilon = O(q)$. H_j^0 is the $q = 0$ limit defined in (4.1) and is diagonal on the paths while H_j^1 has only off-diagonal entries. Examination of (5.4) shows that the action of the off-diagonal part H_j^1 on a given path $\{p_N, \ldots, p_1\}$ is to produce paths $\{p'_N, \ldots, p'_1\}$ with $p'_i = p_i, i \neq j+1, j$, and $p'_{j+1} + p'_j = p_{j+1} + p_j$. In the terminology of [20], each path p' differs from p by an inversion. $|\psi^{(0)}\rangle$ consists of a single path: this was the subject of propositions 1 and 2. Every path in $|\psi^{(k)}\rangle$ differs from $|\psi^{(0)}\rangle$ by at least k inversions.

Now consider the right-hand side of (4.3). It is the sum of contributions from the coefficients of the individual paths in $|\psi'\rangle$ and $|\psi\rangle$. The operator e_{N-1} connects only those paths which differ by ± 1 at the site $N-1$. The non-zero contributions therefore come from just those paths in $|\psi^{(k)}\rangle$ and $|\psi'^{(k')}\rangle$ which are 'linked' in this way. Since the paths $|\psi^{(0)}\rangle$ and $|\psi'^{(0)}\rangle$ agree with the ground state path on sites $j \geqslant M$, and the operators H_j^1 can only create transpositions at adjacent pairs of sites, it is evident that the minimum number of transpositions necessary to make the link is $2N - 2M - 1$, so that the integers k, k' must satisfy $k + k' \geqslant 2N - 2M - 1$. Thus the result is proved for e_0. Obviously the same argument applies to the other Chevalley generators. In this regard, note that the imposition of boundary conditions breaks the exact $U_q(\mathfrak{sl}_2)$ symmetry of $H_{\mathrm{CTM},N}$. □

Proposition 4. Let $\Lambda = k\Lambda_0 + (l-k)\Lambda_1$ be a level-l dominant integral weight with corresponding ground state $\bar{p}_\Lambda$. Choose M and N, $N > 2M + 1$. Let $\{|\psi\rangle\}$ be the set of eigenvectors which correspond, as $q \to 0$, to the crystal base vectors of proposition 2. Then they provide, $\bmod q^{2N-4M-1}$, *a basis of the weight spaces* $\bigoplus_{n=0}^{M} V_n \subset V(\Lambda)$ *in the decomposition* (4.21).

Proof. This follows immediately from propositions 2 and 3. Proposition 2 guarantees the correct identification in the limit $q \to 0$. Proposition 3 ensures that it survives to finite q, $\bmod q^{2N-4M-1}$. □

5. Structure of the eigenvectors and vertex operators

For calculating quantities of physical interest beyond the one-point functions, one must have precise information about the structure of the eigenvectors. Such information follows from the identification of the eigenvectors with weight vectors of irreducible modules via the vertex operator construction. In this section we will give a brief description of the construction and its relation to the finite-size approximation properties of $H_{\mathrm{CTM},N}$.

Let $\Lambda = k\Lambda_0 + (l-k)\Lambda_1$, and write $\Lambda' = (l-k)\Lambda_0 + k\Lambda_1$. The ground state path $\bar{p}_\Lambda \in \mathcal{P}(\Lambda)$ (respectively, $\bar{p}_{\Lambda'} \in \mathcal{P}(\Lambda')$) has $\bar{p}_j = k$ for odd j (respectively, even j) and $\bar{p}_j = (l-k)$ for even j (respectively, odd j). If we diagonalize $H_{\mathrm{CTM},N}$ on a chain of length $N-1$ with the boundary condition that $p_{N-1} = (\bar{p}_{\Lambda'})_{N-1}$, we obtain a finite-size approximation to the module $V(\Lambda')$ on the subspace $W_{\Lambda'} \subset V_l^{\otimes N-1}$ selected by the boundary condition. Now the tensor product $W_{\Lambda'} \otimes V_l$ is the subspace $W_\Lambda \subset V_l^{\otimes N}$ selected by the boundary condition $p_N = (\bar{p}_\Lambda)_N$. So we may diagonalize $H_{\mathrm{CTM},N}$ on this space to obtain a finite-size approximation to $V(\Lambda)$. Moreover, we may use as the basis vectors the tensor products $|\psi\rangle \otimes v_k$, where $|\psi\rangle$ are the previously found eigenvectors and v_k are the basis of V_l. Note that because of the order of the terms in $W_{\Lambda'} \otimes V_l$, it will be necessary

to increment the site labels in the components of $|\psi\rangle$. This construction is the finite-size approximation to the vertex operator construction. Essentially it gives a description of the eigenvectors in which the state at just the first site is made explicit. Unlike the expansion in paths where the state at every site is made explicit, there is no problem in this case in developing a convergent theory for $N \to \infty$.

We turn to the necessary definitions for the vertex operators. These are intertwiners of either $U_q'(\widehat{sl}_2)$ or $U_q(\widehat{sl}_2)$ modules. Here we need the 'type I' vertex operators $\tilde{\Phi}_\lambda^{\mu V} : V(\lambda) \to \widehat{V}(\mu) \otimes V$ which are treated in some detail in [27]. In the case that V is the standard module V_l, we write

$$\tilde{\Phi}_\lambda^{\mu V} : V(\lambda) \to \widehat{V}(\mu) \otimes V \qquad \tilde{\Phi}_\lambda^{\mu V} u_\nu = \sum_{n \in \mathbb{Z}} \sum_{j=0}^{l} u_{\nu - j\alpha_1 + n\delta} \otimes v_j. \tag{5.1}$$

Here $\widehat{V}(\mu)$ is a completion of $V(\mu)$, but we shall not worry about this, simply omitting the hat and assuming the necessary completion. The sum is restricted to those combinations of n and j for which $V(\mu)$ has non-zero weight spaces, and so is bounded above in n. Since $\tilde{\Phi}_\lambda^{\mu V}$ expresses a given weight vector in $V(\lambda)$ in terms of vectors from various weight spaces of $V(\mu)$, we may speak of the weight components $\left(\tilde{\Phi}_\lambda^{\mu V}\right)_{jn} u_\nu$ of the map. The most important results of [27]for our purpose are that the type I vertex operators preserve the crystal base, and that for the level-l $U_q(\widehat{sl}_2)$ modules $V(\Lambda)$, the admissible pairs of highest weights λ, μ are of the form $\Lambda = k\Lambda_0 + (l-k)\Lambda_1$, $\Lambda' = (l-k)\Lambda_0 + k\ ambda_1$, which we already used. The weight component which maps u_Λ to $u_{\Lambda'} \otimes v_k$ has no other entries for this maximum value of n, and we may normalize the vertex operator by choosing its coefficient:

$$\tilde{\Phi}_\Lambda^{\Lambda' V} u_\Lambda = u_{\Lambda'} \otimes v_k + \cdots. \tag{5.2}$$

Together with the intertwining property this uniquely determines $\tilde{\Phi}_\Lambda^{\Lambda' V}$.

We come to the main result of this section. We may regard $\tilde{\Phi}_\Lambda^{\Lambda' V}$ as a matrix with an infinite number of finite-dimensional weight component blocks. It is evident from proposition 4 that the finite-dimensional matrix which effects the basis change needed to diagonalize $H_{\mathrm{CTM},N}$ relative to the basis of $W_{\Lambda'} \otimes V_l$ obtained by using the eigenvectors of $H_{\mathrm{CTM},N-1}$ in $W_{\Lambda'}$, preserves the algebra action, $\bmod q^{2N-4M-1}$. Hence those blocks of the matrix which transform the bases between $\bigoplus_{n=0}^{M} V_n \subset V(\Lambda)$ and $\bigoplus_{n=0}^{M-1} V_n' \otimes V_l \subset V(\Lambda') \otimes V_l$ must be the corresponding finite-dimensional pieces of $\tilde{\Phi}_\Lambda^{\Lambda' V}$, again $\bmod q^{2N-4M-1}$. Formally stated:

Proposition 5. Let Λ, Λ' be level-l dominant integral weights as defined above. Choose M and N, $N > 2M+1$. Let $\{|\psi\rangle\}$, $\{|\psi'\rangle\}$ be the sets of eigenvectors of $H_{\mathrm{CTM},N}$, $H_{\mathrm{CTM},N-1}$ as defined in proposition 4, and use them as the basis vectors of W_Λ and $W_{\Lambda'}$. Consider the change of basis matrix from W_Λ to $W_{\Lambda'} \otimes V_l$. Then, with an appropriate choice of normalization, the entries which connect the finite-dimensional subspaces $\bigoplus_{n=0}^{M} V_n \subset V(\Lambda)$ and $\bigoplus_{n=0}^{M-1} V_n' \subset V(\Lambda')$ are equal, $\bmod q^{2N-4M-1}$, to the entries of the appropriate weight components of $\tilde{\Phi}_\Lambda^{\Lambda' V}$ in the same bases.

6. Connection with Baxter's original arguments for CTMs

In a recent paper [21]the XXZ Hamiltonian spin chain is diagonalized using the weight vectors of the modules $V(\Lambda_0)$, $V(\Lambda_1)$. In particular, the ground state eigenvector is identified with the 'canonical element' of the tensor product $V(\Lambda_0) \otimes V^{*a}(\Lambda_0)$, and thus related to the eigenvectors of the CTM. Baxter's original argument, whereby he derived the remarkable properties of CTMs, also comes from such a construction. In this section we examine the relationship between the two constructions, restricting our attention to the XXZ case.

Baxter considers ([31], p 375) the product of two CTMs which represents an infinite half-plane. He argues that the entries in this product should be, in the limit of large system size, the same as the entries in the maximal eigenvector of the corresponding spin chain for a single row of spins. Let us, in the spirit of Baxter's argument, write this as

$$\Psi = B(\lambda - u)A(u), \tag{6.1}$$

where Ψ is the ground state of the spin chain, A and B are CTMs, u is the spectral parameter, and λ is the crossing parameter. It is related to q by $q = \exp(-\lambda)$. In (6.1) the row and column indices of the matrix product label the spin indices in Ψ to the right and left in the chain. There is no site with the label $j = 0$.

We want to use the exponentiation property $A(u) = \exp(uH)$ to represent the CTMs. (Recall that the ground state of our spin chain has maximal eigenvalue.) The condition for the exponentiation to apply is that the Boltzmann weights satisfy the Yang–Baxter relations and also have the rotational symmetry that the replacement $u \to \lambda - u$ is equivalent to a 90° rotation. The eight-vertex model enjoys these properties, and the six-vertex model is obtained from it by setting two of the weights to zero. There are two possible choices for this, represented by setting $k = 0$ (the elliptic modulus) in either equations (10.4.21) or (13.3.9) of [31]. The former choice is the usual one, used in this paper. However, only the latter choice retains the necessary rotational property. This is discussed in some detail in [28], where it is shown that the two may be related using spin reversal. One must reverse the spins along every second diagonal line, and this can be achieved by similarity transformation using spin reversal operators R_j for site j. Thus we write

$$\begin{aligned} A(\lambda/2) &= \Big(\prod_{j=1}^{\infty} R_{2j}\Big) \exp(uH) \Big(\prod_{j=1}^{\infty} R_{2j}\Big) \\ B(\lambda/2) &= \Big(\prod_{j=1}^{\infty} R_{2j-1}\Big) \exp((\lambda - u)H) \Big(\prod_{j=1}^{\infty} R_{2j}\Big). \end{aligned} \tag{6.2}$$

The reason for the use of R_{2j-1} in one place is that to retain consistency with (6.1), the spin reversal must be for every alternate site of Ψ. The point is that the chosen diagonal direction is different for the two CTMs. We have not yet specified the Hamiltonian generator H. Obviously it is the spin chain H_{XXZ}, after the spin reversal. Namely,

$$H = \Big(\prod_{j=1}^{\infty} R_{2j}\Big) H_{XXZ} \Big(\prod_{j=1}^{\infty} R_{2j}\Big). \tag{6.3}$$

The outcome is that we may transform (6.1) to exponentiated form. We use (2.34) to write it in terms of H_{CTM}:

$$\Psi = R\, q^{2H_{CTM}+S} \qquad R = \prod_{j=1}^{\infty} R_j. \tag{6.4}$$

To complete the argument, we must make an eigenfunction expansion of the right-hand side of (6.4). Thus we write

$$\Psi = R \sum_k q^{2(E_k - E_0)} |\psi_k\rangle\langle\psi_k|. \tag{6.5}$$

In the sum, E_k are eigenvalues of H_{XXZ}, E_0 being the ground-state eigenvalue, while $|\psi_k\rangle$ and $\langle\psi_k|$ are right and left eigenvectors. Recall that the dual space carries a right module representation $V^r(\Lambda)$, and that there is a canonical pairing defined by

$$\langle u_\Lambda | u_\Lambda \rangle = 1 \qquad \langle u | xv \rangle = \langle ux | v \rangle \qquad \forall\, x \in U_q\left(\widehat{sl_2}\right). \tag{6.6}$$

There is also a left module action defined on the dual space via the antipode (2.3e). That is, the left action of x on u^*, denoted xu^*, is given by

$$\langle xu^*, v \rangle = \langle u^*, a(x)v \rangle. \tag{6.7}$$

This is used in [21]to define the dual modules $V^{*a}(\Lambda)$. One sees from section 6.7 of that paper, in particular from the proof of proposition 6.2 therein, that the factor $q^{2(E_k - E_0)}$ together with spin reversal is exactly what is required to replace the left eigenvectors $\langle\psi_k|$ by the corresponding weight vectors u_k^* in the dual module. At the same time, we replace $|\psi_k\rangle$ by u_k. So we conclude that Baxter's construction (6.1) is equivalent to that of [21], namely

$$\Psi = \sum_k u_k u_k^* \tag{6.8}$$

which is the canonical element of $V(\Lambda_0) \otimes V^{*a}(\Lambda_0)$, namely the identity map from $V(\Lambda_0)$ to $V(\Lambda_0)$.

7. Concluding comments

It is evident that the quantum affine symmetry, together with the vertex operator construction, gives a method to construct the eigenvectors of the CTMs and thereby to calculate correlation functions. That is, one may generalize Baxter's trace formula for one-point functions to formulae for n-point functions. Given an operator L acting on the tensor product $V_l^{\otimes N}$, and CTMs $A(u), \cdots, D(u)$ for the four quadrants of the plane lattice, the expectation values of the matrix elements of L are

$$\langle L \rangle_{r_N,\cdots,r_1}^{s_N,\cdots,s_1} = \left[\mathrm{tr}(LABCD)/\mathrm{tr}(ABCD)\right]_{r_N,\cdots,r_1}^{s_N,\cdots,s_1} \tag{7.1}$$

Using the vertex operator construction to expose the dependence of the weight vectors on the local variables at the N chosen sites, and the identification of the CTM generator with the derivation operator d, this translates to the evaluation of certain traces over the representations. Rapid progress is being made in this endeavour [22, 37, 38]; for details, we refer the reader to those papers.

In one of the original papers on CTMs [1], Baxter observes that there is no Ising-like reduction in the six- and eight-vertex models. He wrote:

> *A rather ambitious hope is that by examining the CTMs we may stumble on such a group, that the solution of the models may thereby be simplified ...*

Clearly such a group has been found, at least in the anti-ferromagnetic regime. Moreover, one may confidently expect that this discovery is just the tip of an iceberg. In this paper we have shown that, for certain models associated with $U_q(\hat{sl}_2)$, the quantum affine symmetry conjectured in [20]is indeed an exact infinite-dimensional symmetry. We have also shown that this symmetry may be treated, in a precise way, as the limit of operations on systems of finite size.

In the case of the six-vertex model, the representations needed to evaluate the traces in (7.1) are level-1, and there is a boson construction of the irreducible modules due to Frenkel and Jing [39]. Using it, Jimbo *et al* have given general expressions for the expectation values [22], and have recovered in particular the formula for the spontaneous staggered polarization ($N = 1$). A remarkable feature of this new calculation is that it is the first re-derivation of Baxter's original result [40], despite the endeavours of almost twenty years. The extension of these calculations to $l > 1$ is not trivial, since it depends on having sufficiently tractable presentations of the level-l irreducible modules $V(\Lambda)$.

However, in a very recent work [38], Jimbo *et al* have shown that correlation functions of the type (7.1) satisfy systems of difference equations. One may seek solutions of these equations without explicitly constructing the relevant representations. Moreover, based on a conjecture about the possibility of extending the vertex operator construction to lattice systems which have elliptic parametrisation, these difference equations have been extended to the eight-vertex model. In this way, the conjectured formula [41]for the spontaneous staggered polarization of the eight-vertex model has also been recovered. Unlike the six-vertex case, this is the first derivation of the eight-vertex result. One of the postulates which forms an essential ingredient for the derivation of difference equations is the possibility of using an approximate vertex operator construction on a finite system and then taking the infinite limit. It is one of the results of this paper that such constructions are indeed possible.

In conclusion, let us emphasise that it is *not* a result of this paper that *all* the eigenvectors of a truncated CTM operator approximate weight vectors of a level-l module, with grading levels approximated by the corresponding CTM eigenvalues. This is clearly impossible since all the vectors belong to a finite tensor product of level-0 modules. The statement of proposition 4 gives the correct picture: if one holds fixed the maximum grading level M and the parameter q, then there is convergence as the length N of the chain increases—a purely asymptotic result. For a particular expansion to given order q^L, one might contemplate holding the quantity $L = 2N - 4M - 1$ constant. But then one readily sees that the *fraction* of eigenvectors whose grading level does not exceed M, the only ones which contribute to the chosen order, rapidly diminishes as N increases. This is why the tensor product of N copies of a level-0 module can provide increasingly accurate approximations to the weight vectors of a level-l module: with increasing N, a decreasing fraction of the totality of eigenvectors are involved at any given level of approximation. This non-uniform asymptotic property is almost certainly related to the difficulties in defining the CTMs as renormalized infinite-dimensional operators, or in explicitly constructing their ground state eigenstates. Even if it is not possible to give a satisfactory renormalization scheme in which $H_{\mathrm{CTM},\infty}$ is defined either as $\lim_{N\to\infty}(H_{\mathrm{CTM},N} - E_N(\bar{p}_\Lambda))$, or order by order in a perturbation expansion, the low-lying eigenstates of the truncated CTMs are 'good' eigenstates; they correctly reconstruct the level-l modules with arbitrary precision. And for the computation of any physical quantity as a q-expansion, that is all we need.

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